Period 1, Oct 30, 2024

$\ln \frac{a}{b} = \ln a - \ln b$

Water Bottle Rocket

Students build a rocket that must meet predetermined specifications. At the Olympiad, rockets will be "fueled" with 355 milliliters of water. The rocket with the greatest combined "hang time" and patch design score will be declared the winner. Each school may enter one (1) rocket built by a team consisting of three (3) students. All teams must have: Water-Bottle Vehicle (constructed and launch-ready) and Team Patch.

Cal = 7.37 ⟹ $e^2 = 7.389$

c) $\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x = e^2 = y = \lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x$

$\ln y = \lim_{x\to\infty} \ln\left(1+\frac{2}{x}\right)^x = \lim_{x\to\infty} x \ln\left(1+\frac{2}{x}\right)$

$\frac{\ln\left(1+\frac{2}{\infty}\right)}{\frac{1}{\infty}} = \frac{\ln 1}{\frac{1}{\infty}} = \frac{0}{0}$

$\ln y = \lim_{x\to\infty} \frac{\ln\left(1+\frac{2}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\ln\left(\frac{x+2}{x}\right)}{\frac{1}{x}}$

$\ln y = \lim_{x\to\infty} \frac{\ln(x+2) - \ln(x)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{x}{x}\left(\frac{1}{x+2}\right)\cdot 1 - \frac{1}{x}\frac{(x+2)}{(x+2)}}{\frac{-1}{x^2}} = \lim_{x\to\infty} \frac{\frac{\cancel{x}-\cancel{x}-2}{x(x+2)}}{\frac{-1}{x^2}}$

$\ln y = \lim_{x\to\infty} \frac{\frac{-2}{x(x+2)}}{\frac{-1}{x^2}} = \lim_{x\to\infty} \frac{\cancel{-}2}{\cancel{x}(x+2)} \cdot \frac{x^{\cancel{2}}}{\cancel{-}1} = \lim_{x\to\infty} \frac{2x}{x+2} = \lim_{x\to\infty} \frac{2\cancel{x}}{\cancel{x}} = 2$

$\ln y = 2 \Rightarrow \boxed{e^2 = y}$

b) $\lim_{x\to1^+}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right) = \lim_{x\to1^+}\left[\frac{1(x-1)}{\ln x(x-1)} - \frac{1(\ln x)}{(x-1)(\ln x)}\right]$

$\lim_{x\to1^+}\frac{x-1-\ln x}{(\ln x)(x-1)} = \lim_{x\to1^+}\frac{1-0-\frac{1}{x}}{\frac{1}{x}(x-1)+(\ln x)(1)} = \lim_{x\to1^+}\frac{1-\frac{1}{x}}{1-\frac{1}{x}+\ln x}$

Plug in $1 \Rightarrow \frac{1-1-\ln 1}{(\ln 1)(1-1)} = \frac{0}{0}$

Plug in 1 $\frac{1-\frac{1}{1}=0}{1-\frac{1}{1}+\ln 1=0}$

$\ln 1=0$

$\lim_{x\to1^+}\frac{\frac{x-1}{x}}{\frac{x-1+x\ln x}{x}} = \lim_{x\to1^+}\frac{(x-1)}{\cancel{x}}\cdot\frac{\cancel{x}}{(x-1+x\ln x)} = \lim_{x\to1^+}\frac{1}{1-0+1\cdot\ln x+x\cdot\frac{1}{x}} = 1$

$\lim_{x\to1^+}\frac{1}{1+1+\ln x} = \frac{1}{1+1+\ln 1} = \frac{1}{1+1+0} = \frac{1}{2}$

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b) $y = \lim_{x\to0^+} x^x \Rightarrow y = \lim_{x\to0^+} x^x \Rightarrow \ln y = \lim_{x\to0^+}\ln x^x = \lim_{x\to0^+} x\cdot\ln x$

$0^0 =$

$\ln y = \lim_{x\to0^+}\frac{\ln x}{\frac{1}{x}} = \lim_{x\to0^+}\frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to0^+}\frac{1}{x}\cdot\frac{-x^2}{1} = \lim_{x\to0^+} -x = -0 = 0$

$\frac{\ln(0)}{\frac{1}{0}} = \frac{-\infty}{\infty}$

$\ln y = 0 \Leftrightarrow e^0 = y \Leftrightarrow y=1$

$\lim_{x\to5} x = 5$

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a) $\lim\limits_{x\to\infty} e^{-x}\sqrt{x} = \lim\limits_{x\to\infty} \frac{1}{e^x}\cdot\sqrt{x} = \lim\limits_{x\to\infty} \frac{\sqrt{x}}{e^x} = \lim\limits_{x\to\infty} \frac{x^{\frac{1}{2}}}{e^x} = \lim\limits_{x\to\infty} \frac{\frac{1}{2\sqrt{x}}}{e^x}$

Plug in ∞ $= \frac{\sqrt{\infty}}{e^\infty} = \frac{\infty}{\infty}$

$\Downarrow$

$\lim\limits_{x\to\infty} e^{-x}\sqrt{x} = \lim\limits_{x\to\infty} \frac{\frac{1}{2\sqrt{x}}}{e^x} = \frac{\frac{1}{\infty}}{\infty} = \frac{0}{\infty} = 0$

$0 = \frac{0}{\infty} = \frac{\frac{1}{\infty}}{\infty} = \frac{\frac{1}{2\sqrt{\infty}}}{e^\infty}$

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b) $\lim\limits_{x\to\infty} \frac{3x^2-1}{2x^2+1} \approx \lim\limits_{x\to\infty} \frac{3x^2}{2x^2} = \frac{3}{2}$

$\lim\limits_{x\to\infty} \frac{3x^2-1}{2x^2+1} = \lim\limits_{x\to\infty} \frac{6x}{4x} = \frac{3}{2}$

$\frac{\infty}{\infty}$

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c) $y = \lim\limits_{x\to\infty} x^{\frac{1}{x}} \Rightarrow \ln y = \lim\limits_{x\to\infty} \ln x^{\frac{1}{x}} \Rightarrow \ln y = \lim\limits_{x\to\infty} \frac{1}{x}\cdot\ln x = \lim\limits_{x\to\infty} \frac{\ln x}{x} = \lim\limits_{x\to\infty} \frac{\frac{1}{x}}{1}$

$\infty^{\frac{1}{\infty}} = \infty^0 =$

$\Downarrow$

$\frac{\ln\infty}{\infty} = \frac{\infty}{\infty}$ $\quad 0 = \frac{\frac{1}{\infty}}{1} = \frac{0}{1}$

$\ln y = \lim\limits_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$

$\ln y = 0 \Rightarrow e^0 = y \Rightarrow 1 = y$

b) $$\lim_{x\to 0}\frac{\sqrt{1+x}-1-\frac{x}{2}}{x^2} = \lim_{x\to 0}\frac{\frac{1}{2\sqrt{1+x}}-0-\frac{1}{2}}{2x} = \lim_{x\to 0}\frac{\frac{1}{2\sqrt{1+x}}-\frac{1}{2}}{2x}$$

Plug in 0

$$\frac{\sqrt{1+0}-1-\frac{0}{2}}{0^2} = \frac{\sqrt{1}-1}{0} = \frac{1-1}{0} = \frac{0}{0}$$

$$\frac{\frac{1}{2\sqrt{1+0}}-\frac{1}{2}}{2\cdot 0} = \frac{\frac{1}{2}-\frac{1}{2}}{0} = \frac{0}{0}$$

$$\lim_{x\to 0}\frac{\frac{1}{2}(x+1)^{-\frac{1}{2}}-\frac{1}{2}}{2x} = \lim_{x\to 0}\frac{\frac{1}{2}\cdot-\frac{1}{2}(x+1)^{-3/2}-0}{2} = \lim_{x\to 0}\frac{\frac{-1}{4\sqrt{(x+1)^3}}}{2} =$$

$$-\frac{1}{8} = \frac{\frac{-1}{4\sqrt{(0+1)^3}}}{2} = \frac{\frac{-1}{4\sqrt{1}}}{2} = \frac{\frac{-1}{4}}{\frac{2}{1}}$$

$$y = \frac{1}{2\sqrt{x+1}} - \frac{1}{2} = \frac{1}{2}(x+1)^{-\frac{1}{2}} - \frac{1}{2}$$

$$\frac{dy}{dx} = \frac{1}{2}\cdot-\frac{1}{2}(x+1)^{-\frac{1}{2}-1}\cdot 1 - 0$$

$$\frac{dy}{dx} = \frac{-1}{4}(x+1)^{-\frac{3}{2}} = -\frac{1}{4\sqrt{(x+1)^3}}$$

$$\lim_{x\to\infty}\frac{F(x)}{g(x)} = \frac{\infty}{\infty}$$

$$\lim_{x\to\infty}\frac{F(x)}{g(x)} = \lim_{x\to\infty}\frac{F'(x)}{g'(x)}$$

1.

$$4x^2 + 6x^2y^3 + y^4 = 6x + 2y^3$$

$$8x + 12x\cdot y^3 + 6x^2\cdot 3y^2\frac{dy}{dx} + 4y^3 = 6 \quad + 6y^2\frac{dy}{dx}$$

$$8x + 12xy^3 + \cancel{18x^2y^2\frac{dy}{dx}} + \cancel{4y^3\frac{dy}{dx}} = \cancel{6} \quad + 6y^2\frac{dy}{dx}$$

$$-6 \qquad\qquad -6$$

$$\cancel{-18x^2y^2\frac{dy}{dx}} - \cancel{4y^3\frac{dy}{dx}} \qquad\qquad -18x^2y^2\frac{dy}{dx} - 4y^3\frac{dy}{dx}$$

$$8x + 12xy^3 - 6 = 6y^2\frac{dy}{dx} - 18x^2y^2\frac{dy}{dx} - 4y^3\frac{dy}{dx}$$

$$\frac{8x + 12xy^3 - 6}{6y - 18x^2y^2 - 4y^3} = \frac{\frac{dy}{dx}\cancel{\left(6y^2 - 18x^2y^2 - 4y^3\right)}}{\cancel{6y^2 - 18x^2y^2 - 4y^3}}$$

$$\frac{8x + 12xy^3 - 6}{6y^2 - 18x^2y^2 - 4y^3} = \frac{2(4x + 6xy^3 - 3)}{2(3y^2 - 9x^2y^2 - 2y^3)} = \frac{3 - 4x - 6xy^3}{9x^2y^2 + 2y^3 - 3y^2}$$

$$= \frac{6 - 8x - 12xy^3}{18x^2y^2 + 4y^3 - 6y^2}$$

$$5x^3 + 11x^3y^2 + 2y^3 = 4x + 7y^4$$

$$\cancel{15x^2} + \cancel{33x^2y^2} + 11x^3 \cdot 2y\frac{dy}{dx} + 6y^2\frac{dy}{dx} = 4 + \cancel{28y^3\frac{dy}{dx}}$$

$$-15x^2 \quad -33x^2y^2 \qquad -28y^3\frac{dy}{dx} \qquad \cancel{-28y^3\frac{dy}{dx}} - 33x^2y^2 - 15x^2$$

$$22x^3y\frac{dy}{dx} - 28y^3\frac{dy}{dx} + 6y^2\frac{dy}{dx} = 4 - 33x^2y^2 - 15x^2$$

$$\frac{\frac{dy}{dx}\left(\cancel{22x^3y - 28y^3 + 6y^2}\right)}{\cancel{22x^3y - 28y^3 + 6y^2}} = \frac{4 - 33x^2y^2 - 15x^2}{22x^3y - 28y^3 + 6y^2}$$

$$\frac{dy}{dx} = \frac{-15x^2 - 33x^2y^2 + 4}{22x^3y + 6y^2 - 28y^3}$$

$$4x^2y + 3y^7 = 6$$

$$8xy + 4x^2\frac{dy}{dx} + 21y^6\frac{dy}{dx} = 0$$

$$-8xy \qquad -8xy$$

$$4x^2\frac{dy}{dx} + 21y^6\frac{dy}{dx} = -8xy$$

$$\frac{\frac{dy}{dx}(4x^2+21y^6)}{4x^2+21y^6} = \frac{-8xy}{4x^2+21y^6}$$

$$\frac{dy}{dx} = \frac{-8xy}{4x^2+21y^6}$$

$$\frac{d^2y}{dx^2} = \frac{\left[-8y + -8x\frac{dy}{dx}\right](4x^2+21y^6) - (-8xy)\left(8x+126y^5\frac{dy}{dx}\right)}{(4x^2+21y^6)^2}$$

$$\frac{d^2y}{dx^2} = \frac{\left(-8y-8x\left(\frac{-8xy}{4x^2+21y^6}\right)\right)(4x^2+21y^6) + 8xy\left(8x+126y^5\left(\frac{-8xy}{4x^2+21y^6}\right)\right)}{(4x^2+21y^6)^2}$$

$$\frac{d^2y}{dx^2} = \frac{(-8y)(4x^2+21y^6) + \frac{64x^2y}{\cancel{(4x^2+21y^6)}}\cdot\cancel{(4x^2+21y^6)} + 64x^2y + 1008xy^6\left(\frac{-8xy}{4x^2+21y^6}\right)}{(4x^2+21y^6)^2}$$

$$\frac{d^2y}{dx^2} = \frac{-32x^2y - 168y^7 + 64x^2y + 64x^2y - \frac{8064x^2y^7}{4x^2+21y^6}}{(4x^2+21y^6)^2}$$

$$3xy^2 + 4y^3 = 6$$

$$\cancel{3\cdot y^2} + 3x\cdot 2y\frac{dy}{dx} + 4\cdot 3y^2\frac{dy}{dx} = 0 \quad -3y^2$$

$$\cancel{-3y^2}$$

$$6xy\frac{dy}{dx} + 12y^2\frac{dy}{dx} = -3y^2$$

$$\frac{dy}{dx}\frac{\cancel{(6xy+12y^2)}}{\cancel{(6xy+12y^2)}} = \frac{-3y^2}{(6xy+12y^2)}$$

$$\frac{dy}{dx} = \frac{-3y^2}{(6xy+12y^2)}$$

$$\frac{d^2y}{dx^2} = \frac{-6y\frac{dy}{dx}(6xy+12y^2) - (-3y^2)\cdot\left(6y+6x\frac{dy}{dx}+24y\frac{dy}{dx}\right)}{(6xy+12y^2)^2}$$

$$\frac{d^2y}{dx^2} = \frac{-6y\left[\frac{-3y^2}{\cancel{6xy+12y^2}}\right]\cancel{(6xy+12y^2)} + 9y^2\left[6y+6x\left(\frac{-3y^2}{6xy+12y^2}\right)+24y\left(\frac{-3y^2}{6xy+12y^2}\right)\right]}{(6xy+12y^2)^2}$$

$$18y^3 + 54y^3$$

$$\frac{d^2y}{dx^2} = \frac{-6y\cdot -3y^2 + 54y^3 + 54xy^2\left(\frac{-3y^2}{6xy+12y^2}\right) + \frac{-216y^2}{6xy+12y^2}}{(6xy+12y^2)}$$

$$\frac{dy}{dx} = \frac{-3y^2}{(6xy+12y^2)} = \frac{-\cancel{3}\cdot\cancel{y}\cdot y}{\cancel{3y}(2x+4y)} = \frac{-y}{2x+4y}$$

$$\frac{d^2y}{dx^2} = \frac{-\frac{dy}{dx}(2x+4y) - \left(-y\left(2+4\frac{dy}{dx}\right)\right)}{(2x+4y)^2} = \frac{\frac{y}{\cancel{2x+4y}}\cancel{(2x+4y)} + y\left(2+4\cdot\frac{-y}{2x+4y}\right)}{(2x+4y)^2}$$

$$\frac{d^2y}{dx^2} = \frac{y+2y-\frac{4y^2}{2x+4y}}{(2x+4y)^2} = \frac{3y-\frac{4y^2}{2x+4y}}{(2x+4y)^2}$$

$$\frac{x + (2 + (x-2))}{2x(x+7)}$$

$$\frac{(2x + (x-2))^2}{1 \quad 1}$$